Algebra-I
B. Math - First year
End Semestral Exam 2012-2013

Time: 3hrs
Max score: 100

All questions carry equal marks. Answer any five.

(1) (a) Prove that every group is isomorphic to a subgroup of some symmetric group.
(b) Consider a group G acting on the set of all left cosets of a subgroup H in G by left multiplication. Determine the kernel of the associated permutation representation. Show that this kernel is the largest normal subgroup of G contained in H. 8+12

(2) (a) If G is a finite group and $g_1, g_2, \ldots, g_r$ are representatives of the distinct conjugacy classes in G, with $g_i \notin Z(G)$, $1 \le i \le r$, show that

$$o(G) = o(Z(G)) + \sum_{i=1} i_G(C(g_i))$$

where $C(g_i)$ is the centraliser of g_i in G.
(b) Prove that if $o(G) = p^m$ for some prime p and some integer $m \ge 1$, then G has a nontrivial center. 12+8

(3) (a) Show that a group of order 12 either has a normal Sylow 3 subgroup or is isomorphic to A_4.
(b) Show that if $o(G) = pqr$, where p, q, r are primes with $p < q < r$, then G has a normal Sylow subgroup for either p, q or r. 10+10

(4) (a) Let G' denote the commutator subgroup of a group G. Show that if H is a normal subgroup of G such that G/H is abelian, then $G' \subseteq H$.
(b) Hence show that the commutator subgroup of S_n is A_n. (Hint: A_n is generated by 3-cycles) 10+10

(5) (a) Let G be a group with subgroups H and K such that
(i) H and K are normal in G, and
(ii) $H \cap K = \{e\}$. Show that $HK \cong H \times K$.
(b) Show that if G is a finite abelian group, then G is isomorphic to the (internal) direct product of its Sylow subgroups. 8+12

(6) (a) Let H and K be groups and let $\phi : K \to Aut(H)$ be a homomorphism. Define the *semidirect product* $H \rtimes_\phi K$ of H and K with respect to ϕ.
(b) Show that there exists a non abelian group of order 21. 5+15